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TT_n-maximal digraphs of the minimum size

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Abstract

Let TT_n be the transitive tournament of order n. We say that a digraph D is TT_n-*maximal* if D does not contain TT_n and if the addition of any arc to D results in a digraph that does contain TT_n. We study the problem of finding TT_n-maximal digraphs of order m ($m \geqslant n$) and of the minimum size. The problem is solved for $n=3$ and for $m=n$, $m=n+1$. We state a conjecture relative to the size and the structure of TT_n-maximal digraphs of order m and minimum size.

We consider only finite graphs and digraphs without loops and multiple edges and arcs. Our terminology and notation are standard unless otherwise stated.

A *transitive tournament* (of order n) TT_n is a tournament defined by the following property: if (x,y) and (y,z) are its arcs, then (x,z) is its arc too.

We say that a graph G is H-*maximal* (a digraph D is F-*maximal*) if G (D) does not contain H (F) and if the addition of any edge to G (any arc to D) results in a graph (in a digraph) that does contain H (F).

The most known theorem relative to finding a H-maximal graph is associated with Turán's problem: what is the maximum number of edges in a graph of order m that does not contain the complete graph K_n?

For natural numbers m and q let $T_q(m)$ denote the complete q-partite graph with $\lfloor m/q \rfloor, \lfloor (m+1)/q \rfloor, \ldots, \lfloor (m+q-1)/q \rfloor$ vertices in the various classes. The size of $T_q(m)$ is marked by $t_q(m)$. In 1941 Turán proved the following theorem ([8,9], see also [2, Chapter VI] for details):

Theorem 1. *Let n and m be natural numbers, $n \geqslant 2$. Then every graph of order m and size greater than $t_{n-1}(m)$ contains a K_n. Furthermore, $T_{n-1}(m)$ is the only graph of order m and size $t_{n-1}(m)$ that does not contain a K_n.*

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The theorem above states that the maximum size of K_n-maximal graph of order m equals $t_{n-1}(m)$. The same problem for triangles was considered earlier by Mantel. He proved that a graph of order m and size at least $\lfloor m^2/4 \rfloor + 1$ contains a triangle [5]. The next theorem due to Erdös et al. [4] gives the minimum size of the K_n-maximal graph of order m. In fact, the authors proved a stronger result. Let $k_n(G)$ denote the number of K_n's in a graph G. We say that a graph G is *strongly K_n-maximal* if $k_n(G) < k_n(G^+)$ where G^+ is obtained from G by the addition of an edge.

Theorem 2. *The minimum size of a strongly K_n-maximal ($n \geqslant 3$) graph of order m is $(n-2)(m-n+2) + \binom{n-2}{2}$. The K_n-maximal graph $K_{n-2} + \bar{K}_{m-n+2}$ is the only strongly K_n-maximal graph of order m and the minimum size.*

The digraph analogue of Turán's theorem was solved by Brown and Harary in 1970 [3]. In that paper, the authors consider, in place of K_n, digraphs whose underlying graph is a K_n. In some cases, certain multigraphs are considered too. In particular, they proved the following theorem:

Theorem 3. *Let F be any tournament of order n, $n \geqslant 3$. The maximum size of F-maximal digraph of order m is $2t_{n-1}(m)$. If $F \neq C_3$ then the unique F-maximal digraph of order m and maximum size is a digraph obtained from $T_{n-1}(m)$ by replacing edges by symmetric arcs.*

It is natural to ask about the minimum size of F-maximal digraph for certain digraph F. We restrict our attention to an F being a transitive tournament. Our purpose is to show some results relative to this problem.

From now on, we shall use the following notation: let TT_n denote the transitive tournament of order n and D a TT_n-maximal digraph of order m, $m \geqslant n$. Set $k = m - n$. By $V(D)$ and $A(D)$ we denote the vertex set and the arc set of D, respectively. By $\bar{D}$ we denote the complement of D. A $\bar{D}$ is a digraph such that $V(\bar{D}) = V(D)$ and $(x, y) \in A(\bar{D})$ if, and only if, $(x, y) \notin A(D)$. Let C_l denote the oriented circle of length l, $l \geqslant 3$, i.e. let $V(C_l) = \{v_1, \dots, v_l\}$ and $A(C_l) = \{(v_i, v_{(i+1) \bmod l}),\ i = 1, \dots, l\}$. To shorten notation, by the oriented circle of length 2 we mean a symmetric arc.

For $n = 2$, $D = \bar{K}_m$. For $n = 3$ we found not only TT_3-maximal digraphs but also C_3-maximal digraphs of the minimum size.

Theorem 4. *Let $m \geqslant 3$. The minimum size of a* TT_3*-maximal digraph of order m is equal to $2m - 3$. Only D_1 and D_2* (see Fig. 1) *are* TT_3*-maximal digraphs of order m and the minimum size, where $q \geqslant 1$, $q + p = m - 2$.*

Proof. It is easy to check that D_1 and D_2 are TT_3-maximal digraphs of size $2m - 3$ independently of the numbers p and q. Let D be a TT_3-maximal digraph of size at most $2m - 3$. Observe that each vertex of D has a degree at least 2. Otherwise, the addition of an arc incident with an isolated vertex in D or the addition of an arc

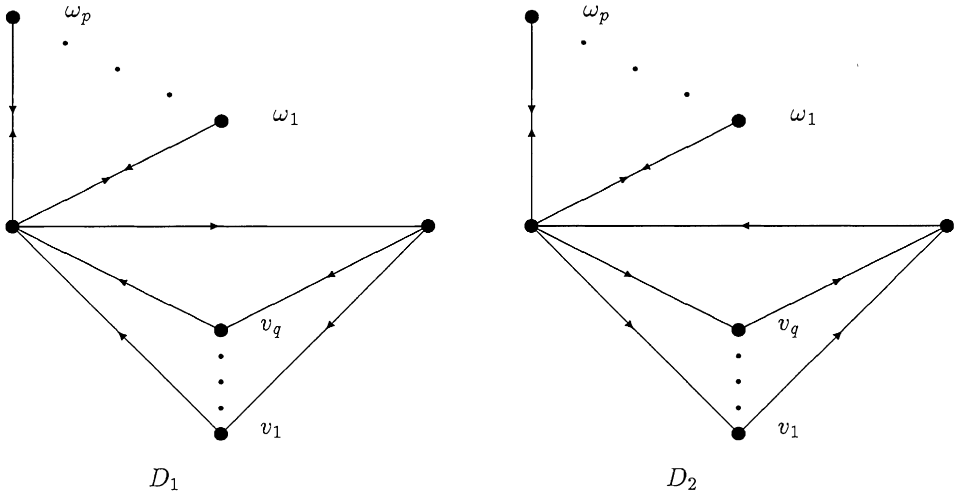

Fig. 1.

orientated opposite to the arc incident with the vertex of degree 1 does not create any tournament of order 3. On the other hand if the size of D is less than or equal to $2m-3$ then the minimum degree of vertices in D is at most 3. Let v be a vertex of D such that v has the minimum degree. Thus $d(v)=2$ or $=3$ and we obtain the following layouts of arcs incident with v.

Layout 1. A vertex u is the only neighbour of v, i.e. $(v,u)\in A(D)$ and $(u,v)\in A(D)$.

Observe that all vertices of D are the neighbours of u. If a vertex v_1 were not the neighbour of u then the addition of an arc (v,v_1) would not create TT_3. We partition the set $V(D)$ into four disjoint subsets. Let $V(D)=\{u\}\cup V_1\cup V_2\cup V_3$ where $V_1=\{v_1\colon (v_1,u)\in A(D)$ and $(u,v_1)\notin A(D)\}$, $V_2=\{v_2\colon (v_2,u)\notin A(D)$ and $(u,v_2)\in A(D)\}$ and $V_3=\{v_3\colon (v_3,u)\in A(D)$ and $(u,v_3)\in A(D)\}$. Set $r_i=|V_i|$, $i=1,2,3$. According to the definition of the TT_3-maximal digraph no two vertices from the same set V_i $(i=1,2,3)$ are neighbours, the only neighbour of vertices from V_3 is u and $(v_1,v_2)\notin A(D)$ for any $v_j\in V_j$, $j=1,2$. On the other hand $(v_2,v_1)\in A(D)$ for $v_j\in V_j$, $j=1,2$. If not, then the addition of such an arc does not create TT_3. Because of that and the size of D, sets V_1 and V_2 are not empty. Now we calculate the number of antisymmetric arcs in D. It is a function f of integers r_1,r_2, $f(r_1,r_2)=r_1+r_2+r_1r_2$ with $r_1,r_2\in[1,m-r_3-2]$ and r_1+r_2 is constant. The function f has the minimum value for $r_1=1$ or $r_2=1$, i.e., when either $D=D_1$ or D_2.

Layout 2. v has two neighbours u_1 and u_2.

Because of the definition of the TT_3-maximal digraph there is the only possibility: $(v,u_1)\in A(D)$, $(u_1,v)\notin A(D)$, $(v,u_2)\notin A(D)$, $(u_2,v)\in A(D)$. It is clear that $(u_2,u_1)\notin$

$A(D)$ and $(u_1,u_2)\in A(D)$ and every vertex of D is a neighbour of u_1 or u_2. Let $V(D)=N_{12}\cup N_1\cup N_2$, where $N_{12}=\{v_{12}$: v_{12} is a neighbour of u_1 and $u_2\}$, $N_1=\{v_1$: v_1 is a neighbour of u_1 and v_1 is not a neighbour of $u_2\}$ and $N_2=\{v_2$: v_2 is not a neighbour of u_1 and v_2 is a neighbour of $u_2\}$. For any vertex $v_{12}\in N_{12}$ $(u_2,v_{12})\in A(D)$, $(v_{12},u_1)\in A(D)$ and $(v_{12},u_2)\notin A(D)$, $(u_1,v_{12})\notin A(D)$. Vertices of N_{12} are nonadjacent. Let $v_1\in N_1$ be joined to u_1 by antisymmetric arc. If $(u_1,v_1)\in A(D)$ then the addition of an arc (v_1,v) does not create TT_3, a contradiction. Let $(v_1,u_1)\in A(D)$. After adding an arc (u_1,v_1), we obtain TT_3. Hence there is a vertex $w\in V(D)\setminus\{v,u_1,u_2\}$ such that it is a neighbour of u_1 and v_1 by antisymmetric arcs (u_1,w) and (w,v_1). Observe that w is not a neighbour of u_2 because TT_3 is not contained in D and $w\notin N_1$ because $(u_1,w)\in A(D)$, which is impossible. The same is applied for vertices of N_2. Hence every vertex $v_i\in N_i$ and u_i are joined by a symmetric arc and then vertices of N_i are nonadjacent, $i=1,2$. Let $v_1\in N_1$ and $v_2\in N_2$. If $(v_1,v_2)\notin A(D)$ we add this arc and conclude that there is $w\in V(D)\setminus\{v,u_1,u_2\}$ such that w,v_1 and v_2 are vertices of C_3 contained in D. Then TT_3 is contained in D also because w is a neighbour of u_1 or u_2. Hence every two vertices $v_1\in N_1$ and $v_2\in N_2$ are joined by a symmetric arc. Observe that vertices of N_{12} are not neighbours of vertices of $N_1\cup N_2$. Now we calculate the number of symmetric arcs in D. It is a function f of integers $r_1=|N_1|$ and $r_2=|N_2|$, $f(r_1,r_2)=r_1+r_2+r_1r_2$ with $r_1,r_2\in[0,r_1+r_2]$ and r_1+r_2 is constant. The function f has the minimum value for $r_1=0$ or $r_2=0$, i.e. when either $D=D_1$ or $D=D_2$.

Layout 3. v has three different neighbours u_1,u_2,u_3.

According to the definition of the TT_3-maximal digraph and the degree of v we obtain that every arc joining v and u_i, $i=1,2,3$ is antisymmetric and $d^+(v)=2$ or $d^-(v)=2$. In both cases if $(v,u_i),(v,u_j)\in A(D)$ or $(u_i,v),(u_j,v)\in A(D)$ then u_i and u_j are not neighbours; if $(v,u_i)\in A(D)$ and $(u_j,v)\in A(D)$ then $(u_i,u_j)\in A(D)$, $(u_j,u_i)\notin A(D)$, $i\neq j$, $i,j\in\{1,2,3\}$. Hence the subdigraph of D indicated by vertices v,u_1,u_2,u_3 has 5 antisymmetric arcs. This subdigraph is TT_3-maximal of the minimum size for $m=4$ but its minimal degree is equal to 2. We may assume that $m>4$. Observe that every vertex of $V(D)\setminus\{v,u_1,u_2,u_3\}$ has to be a neighbour of u_1 or u_2 or u_3. Thus u_1,u_2 and u_3 have together at least $m-4$ neighbours (except of v) such that each of them has a degree at least 3. Then

$$|A(D)|\geqslant 5+\tfrac{1}{2}[m-4+3(m-4)]=2m-3\geqslant|A(D)|.$$

Therefore u_1,u_2 and u_3 have together exactly $m-4$ neighbours (except for v) such that each of them is joined to only one of vertices u_1,u_2,u_3 by an antisymmetric arc. Without loss of generality, we may assume that $d^+(v)=2$ and $(v,u_1)\in A(D)$. The minimal degree in D is equal to 3 so u_1 is a neighbour of at least one vertex different from v,u_2 and u_3. For every $w\in V(D)\setminus\{v,u_1,u_2,u_3\}$, $(u_1,w)\notin A(D)$ because otherwise the addition of (w,v) does not create a TT_3. Hence $(w,u_1)\in A(D)$. We add (u_1,w) and conclude that there is $z\in V(D)\setminus\{v,u_2,u_3\}$ such that u_1,w,z are vertices of C_3 contained in D. Then $(u_1,z)\in A(D)$, a contradiction. □

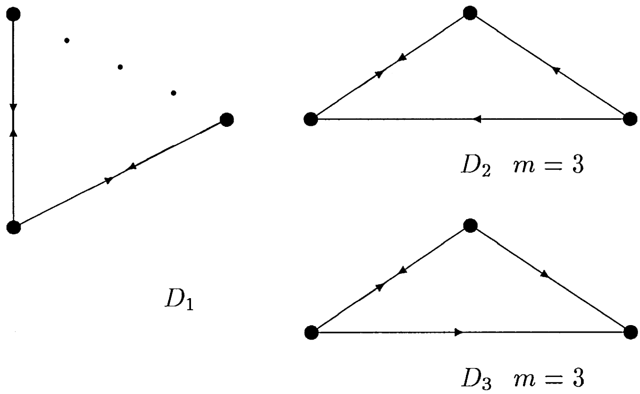

Fig. 2.

Theorem 5. *Let $m \geqslant 3$. The minimum size of a C_3-maximal digraph of order m is equal to $2m - 2$. Only digraphs D_1 for $m \geqslant 3$ and D_2, D_3 for $m = 3$ are C_3-maximal digraphs of the minimum size* (*see* Fig. 2).

Proof. We use the same method as in the proof of Theorem 4. It is easy to see that D_1, D_2 and D_3 are C_3-maximal digraphs of size $2m-2$. Let D be a C_3-maximal digraph of size at most $2m - 2$. Then minimum degree of vertices in D is equal to 2 or 3. Let v be a vertex of the minimum degree. We consider layouts of arcs incident with v.

Layout 1. v has only one neighbour u, i.e. $(v,u),(u,v) \in A(D)$.

Observe that every vertex of D is a neighbour of u. For any $w \in V(D) \setminus \{u\}$ $(w,u) \in A(D)$ and $(u,w) \in A(D)$ because otherwise we add (w,v) or (v,w), respectively, and do not obtain C_3. Hence in this case $D = D_1$.

Layout 2. $d(v) = 2$ and v has two neighbours u_1 and u_2 i.e. v and u_i are joined by antisymmetric arcs, for $i = 1, 2$.

Assume first that $(v,u_1) \in A(D)$ and $(u_2,v) \in A(D)$. We add (u_1,v) and we do not obtain C_3 independently of the existence of arcs joining u_1 and u_2. Hence we have one of following two cases: either $(v,u_1),(v,u_2) \in A(D)$ or $(u_1,v),(u_2,v) \in A(D)$. In both of them u_1 and u_2 are joined by a symmetric arc. If $m=3$ we obtain D_2 or D_3. Let $m > 3$ and let $w \in V(D) \setminus \{v,u_1,u_2\}$. We add (v,w) in the first case or (w,v) in the second one and we do not create C_3. Hence, for this layout $m \geqslant 4$ is impossible.

Layout 3. $d(v) = 3$ and v has two neighbours u_1 and u_2.

Without loss of generality, we may assume that $(v,u_1),(u_1,v) \in A(D)$ and $(v,u_2) \in A(D)$, $(u_2,v) \notin A(D)$. It is easy to check that $(u_1,u_2) \in A(D)$ and $(u_2,u_1) \notin A(D)$.

For $m=3$ we obtain D_3. Let $m>3$. Observe that any vertex of D is a neighbour of u_1 or u_2. Hence the subdigraph of D indicated by vertices v, u_1 and u_2 has four arcs and u_1 and u_2 have together $m-3$ neighbours (except for v) such that each of them has a degree at least three. Therefore,

$$|A(D)| \geqslant 4 + \tfrac{1}{2}[m-3+3(m-3)] = 2m-2 \geqslant |A(D)|.$$

The equality holds if every vertex w of $V(D)\backslash\{v, u_1, u_2\}$ is a neighbour of only one of vertices u_1 or u_2 and vertices w, u_i are joined by an antisymmetric arc for $i=1,2$. This case is impossible because for $i=1,2$ if $(w, u_i)\in A(D)$ we add (w, v), if $(u_i, w)\in A(D)$ we add (v, w) and we do not obtain a C_3.

Layout 4. v has three different neighbours u_1, u_2 and u_3.

Arcs joining v and u_i arc antisymmetric, $i=1,2,3$. We will show that there are no two arcs (v, u_i), $(u_j, v)\in A(D)$, $i \neq j$, $i, j\in\{1,2,3\}$. Conversely, suppose that (v, u_1), $(u_2, v)\in A(D)$. We add (u_1, v) and if (v, u_3), $(u_3, u_1)\in A(D)$, we obtain C_3. Then after adding (u_3, v) we do not obtain C_3, a contradiction. Hence either $d^+(v)=3$ or $d^-(v)=3$. We use the same arguments like in Layout 2 to prove that it is impossible that $m>4$. It is easy to check that we obtain a C_3-maximal digraph in two cases: arcs joining vertices u_1, u_2 and vertices u_2, u_3 are symmetric but vertices $u_1$and u_3 are not neighbours in D; and if $u_1$and u_2 are joined by a symmetrical arc and (u_1, u_3), $(u_2, u_3)\in A(D)$, (u_3, u_1), $(u_3, u_2)\notin A(D)$. The above-described digraphs have size greater than the minimum size for $m=4$. So, for this configuration we do not obtain a C_3-maximal digraph of the minimum size. □

Now look at the problem of finding TT_n-maximal digraphs of the minimal size from the another point of view. Let us study the number $k=m-n$. The problem is solved for $k=0$ and 1. We also obtain necessary and sufficient condition for a digraph to be TT_n-maximal of the minimal size for any k. Let $k=0$, i.e. we look for a TT_n-maximal digraph of order n and of the minimum size. Let us recall two well-known propositions.

Proposition 1 (Moon [6]). *A tournament is transitive if, and only if, it is acyclic.*

Proposition 2 (Rédei [7]). *Every tournament contains a Hamiltonian path.*

Proposition 3. *Let D be a digraph of order n. Then D does not contain* TT_n *as a subdigraph if, and only if, $\bar{D}$ contains a circle C_l with $l\geqslant 2$.*

Proof. By Proposition 1 it is sufficient to prove that if D does not contain TT_n then $\bar{D}$ contains an oriented circle. Let us suppose that $\bar{D}$ does not contain any oriented circles. We shall prove that then D contains TT_n as a subdigraph. The proof is by induction on n. The case when $n=3$ is left for the reader. Let $n\geqslant 4$. Observe that there is a vertex v of D such that every arc incident with v is symmetric or has a form (v, w),

$w \in V(D)$. By the induction hypothesis the subdigraph of D indicated by vertices of $V(D) \setminus \{v\}$ contains TT_{n-1}. Then D contains TT_n such that v is a vertex of outdegree $n-1$ in TT_n. $\square$

Hence we obtain:

Theorem 6. *A digraph D of order $n \geqslant 3$ is TT_n-maximal if, and only if, $\bar{D} = C_l$ $(2 \leqslant l \leqslant n)$.*

Corollary 1. *A digraph D of order n is TT_n-maximal of the minimum size if, and only if, $\bar{D} = C_n$.*

Corollary 2. *The minimum size of a TT_n-maximal digraph of order n is equal to $n(n-2)$.*

Observe that by Proposition 2 the addition of any arc to a TT_n-maximal digraph of order n results in a digraph that does contain any tournament of order n.

By Theorem 6 it is easy to check that:

Corollary 3. *The digraph D of order $m = n + k$ is TT_n-maximal if, and only if the following conditions hold*:

(i) *for any set $W \subset V(D)$, $|W| = n$, the subdigraph of $\bar{D}$ induced by vertex-set W contains an oriented circle,*

(ii) *for any arc $(x, y) \in A(\bar{D})$ there exists set $W \subset V(D)$, $|W| = n$ such that the subdigraph of $\bar{D}$ induced by vertex-set W with removed arc (x, y) does not contain any oriented circles.*

In fact, by the corollary above, if D is TT_n-maximal digraph then every arc of $\bar{D}$ has to be an arc of an oriented circle of $\bar{D}$. Digraph D is a TT_n-maximal digraph of the minimum size if, and only if, D satisfies conditions (i) and (ii) of Corollary 3 and $\bar{D}$ has the maximum number of arcs.

Proposition 4. *Let D be a TT_n-maximal digraph of order m, $m \geqslant 3$ and minimum size. Then $\bar{D}$ does not have isolated vertices.*

Proof. Let D be a TT_n-maximal digraph of order $m = n + k$ and minimum size. Let us suppose that v is an isolated vertex of $\bar{D}$. By condition (i) of Corollary 3 there is an oriented circle C in $\bar{D}$. Let (v_1, v_2) be an arc of C. We remove arc (v_1, v_2) and add arcs (v_1, v), (v, v_2). In this way we obtain a digraph D' such that $V(D') = V(D)$ and $|A(\bar{D}')| = |A(\bar{D})| + 1$. Using Corollary 3 we shall show that D' is a TT_n-maximal digraph. Let $W \subset V(D')$, $|W| = n$. If $v \in W$ then the subdigraph of D' induced by W contains an oriented circle because the subdigraph of D induced by W contains an oriented circle. Let $v \notin W$. Let us suppose that the subdigraph of D' induced by W does not contain any oriented circles. Then the subdigraph of D induced by $(W\setminus\{v\})\cup\{v_1\}$ does not contain any oriented circles, a contradiction. It is easily seen that D' satisfies condition (ii) of Corollary 3. Hence we obtain a contradiction. □

Let D be a TT_n-maximal digraph of order m. By Proposition 4 we may assume that $\bar{D}$ does not have isolated vertices. Observe that, under this assumption, the size of $\bar{D}$ is equal to m if $\bar{D}$ is a layout of $k+1$ disjoined oriented circles. The size of $\bar{D}$ is greater than m if in $\bar{D}$ there are oriented circles with common vertices.

Proposition 5. *Let D be a TT_n-maximal digraph of order m, $m\geqslant 3$. Let $v\in V(D)$ such that v is not an isolated vertex of $\bar{D}$. Then there are v_1, $v_2\in V(D)$ such that (v_1,v), $(v,v_2)\in A(\bar{D})$.*

Proof. It is obvious that every no isolated vertex of $\bar{D}$ has a degree at least 2 in $\bar{D}$. If v is incident with a symmetric arc in $\bar{D}$ then as $v_1=v_2$ we can set its neighbour. Hence we may assume that v has at least two different neighbours in $\bar{D}$ and v is not incident with a symmetric arc in $\bar{D}$. To obtain a contradiction, suppose, without loss of generality, that for every neighbour v_1 of v in $\bar{D}$ $(v,v_1)\in A(\bar{D})$. By condition (ii) of Corollary 3 there is a set $W\subset V(D)$, $|W|=n$ such that a subdigraph of $\bar{D}$ indicated by vertex-set W with removed arc (v,v_1) does not contain any oriented circle. It is easily seen that a subdigraph of $\bar{D}$ indicated by W does not contain any oriented circle as well, which contradicts condition (i) of Corollary 3. □

Definition 1. Let D be a TT_n-maximal digraph of order $m=n+k$, $n\geqslant 3$, $k\geqslant 0$. Let $v\in V(D)$ such that the degree of v in $\bar{D}$ is equal to 2 and v is not incident with a symmetric arc in $\bar{D}$. Then we say that D' is a digraph obtained from digraph D by the *contraction of the first type* if $V(\bar{D}')=V(\bar{D})\setminus\{v\}$ and $A(\bar{D}')=(A(\bar{D})\setminus\{(v_1,v),(v,v_2)\})\cup\{(v_1,v_2)\}$, where v_1, $v_2\in V(\bar{D})$ are neighbours of v in $\bar{D}$ such that (v_1,v), $(v,v_2)\in A(\bar{D})$.

Remark. Observe that if D' is obtained from D by the contraction of the first type then $|V(D')|=|V(D)|-1$ and $|A(\bar{D}')|=|A(\bar{D})|-1$. Note that, by Corollary 3, if D is TT_n-maximal digraph and (v_1,v), $(v,v_2)\in A(\bar{D})$ then $(v_1,v_2)\notin A(\bar{D})$.

Proposition 6. *Let D be a TT_n-maximal digraph of order $m=n+k$, $n\geqslant 3$, $k\geqslant 0$. Let D' be a digraph obtained from digraph D by the contraction of the first type. Then D' is a TT_{n-1}-maximal digraph of order $m-1=(n-1)+k$. Moreover, if D is a TT_n-maximal digraph of minimum size then D' is a TT_{n-1}-maximal digraph of minimum size.*

Proof. It is easy to check that D' satisfies conditions (i) and (ii) of Corollary 3. Let D be a TT_n-maximal digraph of minimum size. Let us suppose that D' is a TT_{n-1}-maximal digraph and D' does not have minimum size. Let D'' be a TT_{n-1}-maximal digraph of order $m-1$ and minimum size. Let $(v_1,v_2)\in A(\bar{D}'')$ and $v\notin V(D'')$. By D''' we denote a digraph of order m such that $V(D''')=V(D'')\cup\{v\}$ and $A(\bar{D}''')=(A(\bar{D}'')\setminus\{(v_1,v_2)\})\cup\{(v_1,v),(v,v_2)\}$. Using the similar method as in proof of Proposition 4 we obtain that D''' is a TT_n-maximal digraph. Moreover, $|A(\bar{D}''')|=|A(\bar{D}'')|+1<|A(\bar{D}')|+1=|A(\bar{D})|$, which contracts the fact that D is a TT_n-maximal digraph of minimum size. □

Definition 2. Let D be a TT_n-maximal digraph of order $m=n+k$, $n\geqslant 3$, $k\geqslant 0$. Let $v\in V(D)$ such that the degree of v in $\bar{D}$ is greater than 2 and v is not incident with a symmetric arc in $\bar{D}$. Assume that there is a neighbour w of v in $\bar{D}$ such that v, w and an arc joining v and w are contained by the same circles in $\bar{D}$. Then we say that D' is a digraph obtained from D by the *contraction of the second type* if $V(D')=V(D)\setminus\{v\}$ and $A(\bar{D}')=(A(\bar{D})\setminus\{(v,u),(u,v)\colon u\in V(D)\})\cup\{(w,u)\colon (v,u)\in A(\bar{D}), u\neq w\}\cup\{(u,w)\colon (u,v)\in A(\bar{D}), u\neq w\}$.

Remark. Observe that if D' is obtained from D by the contraction of the second type then $|V(D')|=|V(D)|-1$ and $|A(\bar{D}')|=|A(\bar{D})|-1$.

Using the same method as in proof of Proposition 6 we obtain:

Proposition 7. *Let D be a TT_n-maximal digraph of order $m=n+k$, $n\geqslant 3$, $k\geqslant 0$. Let D' be a digraph obtained from digraph D by the contraction of the second type. Then D' is a TT_{n-1}-maximal digraph of order $m-1=(n-1)+k$. Moreover, if D is a TT_n-maximal digraph of minimum size then D' is a TT_{n-1}-maximal digraph of minimum size.*

Let us define a set of digraphs:

Definition 3. Let K^*_{k+2} denote a digraph obtained from K_{k+2} by replacing edges by symmetric arcs. Fix $k\geqslant 0$. Then we say that a digraph D is a digraph of set LC_k if, and only if, either $\bar{D}=K^*_{k+2}$ or K^*_{k+2} can be obtained from $\bar{D}$ by a sequence of contractions of the first and the second type.

It is easily seen that:

Proposition 8. *If $D\in LC_k$ and $|V(D)|=n+k$, $n\geqslant 2$ then D is a TT_n-maximal digraph.*

For the describing of a special type of TT_n-maximal digraphs we need the following definitions.

Definition 4. Let v be a vertex of a digraph D and let there are two different neighbours v_1,v_2 of v in D such that (v,v_1) and $(v,v_2)\in A(D)$ or (v_1,v) and $(v_2,v)\in A(D)$. Without loss of generality let us suppose that (v,v_1), $(v,v_2)\in A(D)$. Let $w\notin V(D)$. Then we say that a digraph D' is obtained from D by *gluing* if $V(D')=V(D)\cup\{w\}$ and $A(D')=(A(D)\setminus\{(v,v_1),(v,v_2)\})\cup\{(v,w),(w,v_1),(w,v_2)\}$. The arc (v,w) is called the glued arc.

Definition 5. Let KT^*_{k+2}, $k \geqslant 0$, denote a digraph obtained from K^*_{k+2} by a sequence of gluings and satisfies the following conditions:

(a) KT^*_{k+2} satisfies conditions (i) and (ii) of Corollary 3 for $n = |V(KT^*_{k+2})| - k$,

(b) the set of glued arcs of KT^*_{k+2} creates $k+2$ disjoined trees (including a simple vertex as a tree of order 1),

(c) it is impossible to obtain any digraph from KT^*_{k+2} by the contraction of the first or of the second type.

Definition 6. Fix $k \geqslant 0$. Then we say that a digraph D is a digraph of set LCT_k if, and only if, either $\bar{D} = KT^*_{k+2}$ or KT^*_{k+2} can be obtained from $\bar{D}$ by a sequence of contraction of the first and the second type.

Corollary 4. *For any* $k \geqslant 0$: $LC_k \subset LCT_k$ *and* $LC_1 = LCT_1$.

Proposition 9. *If* $D \in LCT_k$ *and* $|V(D)| = n+k$, $n \geqslant 2$ *then* D *is a* TT_n*-maximal digraph.*

Corollary 5. *Let* $n \geqslant 3$. *If* D *is a* TT_n*-maximal digraph of minimum size then there are* n *vertices of* D *such that every two of them are neighbours.*

Proof. Let us suppose that there are not n vertices of D such that every two of them are neighbours. Then every arc of D is symmetric. Under this condition, D has minimum size if D is obtained from K_n-maximal graph of minimum size by replacing edges by symmetric arcs (see Theorem 2). Observe that for any digraph $D' \in LC_k$, $|A(D')| < |A(D)|$, a contradiction. $\square$

In the next theorem we consider TT_n-maximal digraph of order $n+1$ and of minimum size.

Theorem 7. *Let* $n \geqslant 3$. *The minimum size of* TT_n*-maximal digraph of order* $n+1$ *is equal to* $n^2 - 4$. *Only digraphs on* $n+1$ *vertices of set* LC_1 *are* TT_n*-maximal digraphs of order* $n+1$ *and minimum size.*

Proof. By Proposition 8, any digraph on $n+1$ vertices of LC_1 is TT_n-maximal digraph of size $n^2 - 4$. Using Corollary 3 we shall prove that it is minimum size. Let D be a TT_n-maximal digraph of order $n+1$ and minimum size. By Propositions 6 and 7 we may assume that it is impossible to obtain any digraph from D by the contraction of the first or of the second type. In this way, we simplify the description of the structure of D.

In the proof we will consider $\bar{D}$. By Proposition 4 there are no isolated vertices in $\bar{D}$.

The proof is divided into a sequence of observations and cases.

Observation 1. $\bar{D}$ contains at least two different oriented circles.

We choose two the shortest different circles $C'$and C'' of $\bar{D}$.

Observation 2. If C' and C'' do not have any common vertices then $\bar{D}$ consists of only C' and C''.

Observe that in this case D is a TT_n-maximal digraph but D does not have minimum size, a contradiction. Hence we may assume that C' and C'' have common vertices. To shorten the notation a common single vertex will be called a common path of length 0.

In the following cases we consider all possible situations.

Case 1. Circles $C'$and C'' have at least three pairwise disjoined common paths and arcs of $C'$and C'' create at least one circle which does not contain all common paths of C' and C''.

In fact, there are at least three different circles B_1, B_2, B_3 created by arcs of C' and C''. Each of them does not contain at least one common path of C' and C'' and there is not a common vertex of B_1, B_2 and B_3. Because C' and C'' are the shortest circles, circles B_1, B_2 and B_3 have another vertices. These vertices are common vertices of B_i $(i=1,2,3)$ and circles different from C' and C''. Hence, there is at least one arc of $\bar{D}$ which is not an arc of C' or C''. For this arc we need at least two vertices such that $\bar{D}$ without this arc and these vertices do not contain any oriented circles, which contradicts condition (ii) of Corollary 3.

Hence, we can assume that if C' and C'' have at least two disjoint common paths then every circle created by arcs of C' and C'' contains all common paths of C' and C''.

Case 2. Circles C' and C'' have exactly two disjoint common paths.

Let $P_1=(x_s,\dots,x_e)$ and $P_2=(y_s,\dots,y_e)$, $x_e\neq y_s$, $y_e\neq x_s$, be common paths of C' and C''. It is possible that $x_s=x_e$or $y_s=y_e$. By $E_1=(x_e,\dots,y_s)$ and $E_2=(y_e,\dots,x_s)$ we denote the remaining directed paths creating with P_1 and P_2 the circle C'. By $E_3=(y_e,\dots,x_s)$ and $E_4=(x_e,\dots,y_s)$ we denote the remaining directed paths creating with P_1 and P_2 the circle C''. Let e_i be an arc of E_i $(i=1,2,3,4)$ such that e_i has one common vertex with the path P_1 (see Fig. 3). According to condition (ii) of Corollary 3, for every arc e_i, $i=1,2,3,4$, by a_i, respectively, we denote a vertex such that a digraph obtained from $\bar{D}$ by removing e_i and a_i does not contain any oriented circles. Observe that every a_i is either a common vertex of C' and C'' or is an internal vertex of suitable path E_j, $j\in\{1,2,3,4\}$.

We shall consider two cases:

(2.1) Every vertex a_i is an internal vertex of suitable path E_j, $i,j\in\{1,2,3,4\}$.

(2.2) At least one of vertices a_i, $i\in\{1,2,3,4\}$, is a vertex of P_1 or P_2.

(2.1) For $i=1,2,3,4$, $a_i\neq x_s$, $a_i\neq y_s$, $a_i\neq x_e$, $a_i\neq y_e$. It is clear that $a_1\in E_4$, $a_2\in E_3$, $a_3\in E_2$ and $a_4\in E_1$.

By condition (i) of Corollary 3 there is a circle C''' in $\bar{D}$ such that x_s is not the vertex of C'''. Observe that e_2 and e_3 are not arcs of C''' and that only one of arcs e_1, e_4 can be an arc of C'''.

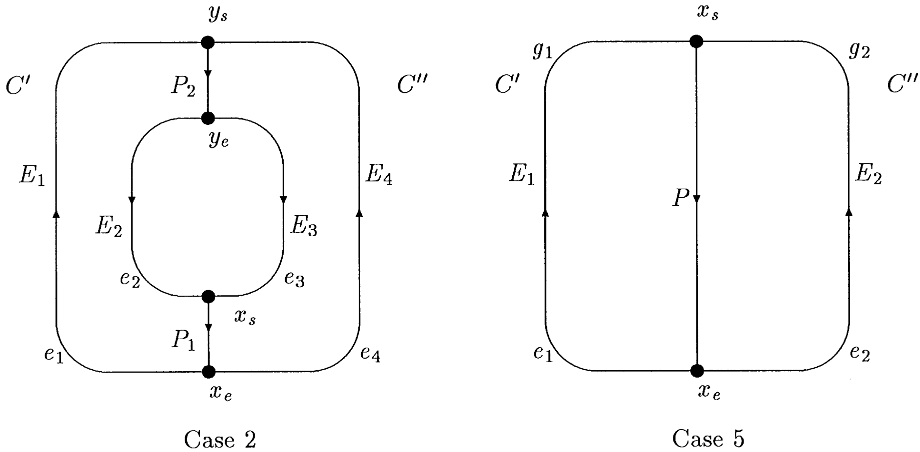

Fig. 3.

(2.1.1) Let us first suppose that C''' does not contain any arcs e_i, $i=1,\dots,4$. Then C''' contains every vertex a_i, $i=1,\dots,4$. Vertices a_1, a_2, a_3, a_4 divide circle C''' into four directed paths: $(w_{js},\dots,w_{je})$, $j=1,2,3,4$ such that each of these paths contains exactly two vertices of $\{a_1,a_2,a_3,a_4\}$ and $w_{js},w_{je}\in\{a_i,\ i=1,2,3,4\}$, $j=1,2,3,4$. Observe that there is one of such four paths: $(a_2,\dots,a_1)$, $(a_2,\dots,a_4)$, $(a_3,\dots,a_1)$, $(a_3,\dots,a_4)$. If there is $(a_2,\dots,a_1)$ or $(a_2,\dots,a_4)$ then in a digraph obtained from $\bar{D}$ by removing e_3 and a_3 there is a circle created by paths: $(a_2,\dots,a_1)$ (or $(a_2,\dots,a_4)$), the part of E_4 from a_1 to y_s (or the part of E_1 from a_4 to y_s), P_2 and the part of E_3 from y_e to a_2. If there is $(a_3,\dots,a_1)$ or $(a_3,\dots,a_4)$ then in a digraph obtained from $\bar{D}$ by removing e_2 and a_2 there is a circle created by paths: $(a_3,\dots,a_1)$ (or $(a_3,\dots,a_4)$), the part of E_4 from a_1 to y_s (or the part of E_1 from a_4 to y_s), P_2 and the part of E_2 from y_e to a_3. This contradicts condition (ii) of Corollary 3.

(2.1.2) Without loss of generality, we may suppose that e_1 is an arc of C'''. Circle C''' contains vertices a_2, a_3 and a_4. We use similar arguments as in case (2.1.1) and obtain a contradiction.

(2.2) Without loss of generality, we can assume that a_1 is a common vertex of C' and C''. By condition (i) of Corollary 3 there is a circle C''' in $\bar{D}$ such that a_1 is not a vertex of C'''. Hence e_1 is an arc of C''' and $a_1\neq x_e$. Circle C''' does not contain internal vertices of E_4 and also y_s is not a vertex of C''' because otherwise there is an oriented circle in a digraph obtained from $\bar{D}$ by removing e_1 and a_1. Moreover, arc e_4 is not an arc of C''' and at most one of arcs e_2, e_3 can be an arc of C'''. Let path $F_1=(x_f,\dots y_s)$ be the part of the path E_1 such that x_f is the only common vertex of C''' and F_1. Without loss of generality we may assume that for any circle C such that a_1 is not a vertex of C, common vertices of C and E_1 are not internal vertices of F_1. Also, without loss of generality we can assume that for every arc (u,w) of E_1, if (u,w) is not an arc of F_1, then (u,w) is an arc of C'''. By f_1 we denote the arc of F_1 such that f_1 is incident with vertex x_f. By condition (ii) of Corollary 3 for f_1 by b_1 we denote a vertex such that a digraph obtained from $\bar{D}$ by removing f_1 and b_1 does not contain any oriented circle. Vertex b_1 is a vertex of E_4 or P_1 or P_2. Arc f_1 is not an arc of C''' and then b_1 is a vertex of C'''. Hence b_1 is a vertex of P_1 or P_2. By condition (i) of Corollary 3 there is a circle C^{iv} such that b_1 is not a vertex of C^{iv} and then f_1 is an arc of C^{iv}. Under the assumption of F_1, arc e_1 is not an arc of C^{iv} and then a_1 is a vertex of C^{iv}. At most one arc of e_2, e_3 can be an arc of C^{iv}. If one of arcs e_2, e_3 are not a common arc of C''' and C^{iv} then a_3 or a_2, respectively, is a common vertex of C''' and C^{iv}. If each of circles C''', C^{iv} contains one of arcs e_2, e_3 then x_s is a common vertex of C''' and C^{iv}. Hence C''' and C^{iv} have common vertices apart from v_f. These common vertices are vertices of paths P_1 or P_2 or E_1 or E_2. Let w be one of these common vertices. Arcs of C''' and C^{iv} create another circles. Under the assumption of C''', one of these circles does not contain arcs e_1 and f_1. We denote it by C^v. It is clear that a_1, b_1 are vertices of C^v. Let v be a common vertex of C''' and one of paths P_2, E_2, E_3 or P_1. Let $(v,\dots,x_e)$ be a directed path such that each arc is an arc of P_1, P_2, E_2 or E_3. If a_1 is a vertex of such a path $(v,\dots,x_e)$ then a digraph obtained from $\bar{D}$ by removing a_1 and e_1 contains an oriented circle, a contradiction. Observe that circle C^v is created by the following directed paths: $A_1=(x_f,\dots,v)$, $A_2=(v,\dots,w)$, $A_3=(w,\dots,x_f)$, such that each arc of A_1 is an arc of C''', each arc of A_2 is an arc of C' or C'' and each arc of A_3 is an arc of C^{iv}. Hence b_1 is a vertex of A_2 and a_1 is a vertex of A_3. We consider the following three circles: The first circle is created by directed paths: $A_4=(x_e,\dots,x_f)$, A_1, $A_5=(v,\dots,x_e)$, such that each arc of A_4 is an arc of E_1, each arc of A_5 is an arc of C' or C''. The second circle is created by directed paths: F_1, $A_6=(y_s,\dots,a_1)$, $A_7=(a_1,\dots,x_f)$, such that each arc of A_6 is an arc of C' or C'', each arc of A_7 is an arc of A_3. The third circle is created by directed paths: $A_8=(a_1,\dots,v)$, A_2, $A_9=(w,\dots,a_1)$ such that each arc of A_8 is an arc of C' or C'', each arc of A_9 is an arc of A_3. Observe that there is not a common vertex of these three circles and there is an arc of C' or C'' which is not an arc of above-described circles. Using condition (ii) of Corollary 3 we obtain a contradiction.

Case 3: Circles C' and C'' have at least three pairwise disjoined common paths. Arcs of C' and C'' create circles such that each of these circles contains all common paths of C' and C''.

Let C' and C'' have p pairwise disjoined common paths $P_i=(x_{si},\dots,x_{ei})$, $i=1,\dots,p$. Observe that each of circles C', C'' consists of paths P_i and directed paths from x_{ei} to $x_{s(i+1)_{\mathrm{mod}\ p}}$, $i=1,\dots,p$. We choose P_1, P_2 and paths (parts of C', C'') from x_{e1} to x_{s2} and from x_{e2} to x_{s3}. For these paths we repeat the arguments of Case 2 and obtain a contradiction.

Hence we may assume that the circles C' and C'' have only one common path. By the condition (i) of Corollary 3 there is at least one arc in $\bar{D}$ such that it is not an arc of C' or C''.

Case 4: Circles C' and C'' have exactly one common path. There is a circle C''' in $\bar{D}$ such that no vertex of common path of C' and C'' is a vertex of C'''.

By condition (ii) of Corollary 3 C''' and C' have a common vertex and C''' and C'' have a common vertex. Moreover, there are no more arcs in $\bar{D}$ and C''' and C' have only one common path, C''' and C'' have only one common path. Then $D \in LC_1$.

Case 5: Circles C' and C'' have exactly one common path P. Every circle of $\bar{D}$ contains at least one vertex of the path P.

We use similar method as in Case 2. It is clear that P has length greater than 0. Set $P=(x_s,\dots,x_e)$, $x_s \neq x_e$. By $E_1=(x_e,\dots,x_s)$ we denote the remaining part of C'. By $E_2=(x_e,\dots,x_s)$ we denote the remaining part of C''. Let e_i be an arc of E_i incident with vertex x_e and g_i be an arc of E_i incident with vertex x_s, $i=1,2$ (see Fig. 3). According to condition (ii) of Corollary 3 by a_i, respectively, we denote a vertex such that a digraph obtained from $\bar{D}$ by removing e_i and a_i does not contain any oriented circles, $i=1,2$. Vertex a_1 is a vertex of P or E_2, vertex a_2 is a vertex of P or E_1.

(2.1) Assume first that a_1 and a_2 are internal vertices of E_2 and E_1, respectively. By condition (i) of Corollary 3 there is a circle C''' such that x_e is not a vertex of C'''. Thus arcs e_1 and e_2 are not arcs of C'''. Hence a_1 and a_2 are vertices of C'''. Let v be a common vertex of C''' and P. Without loss of generality, we may assume that v is a vertex of path $(a_1,\dots,a_2)$, such that its arcs are arcs of C'''. Let us consider a circle created by the following paths: $A_1=(v,\dots,a_2)$, $A_2=(a_2,\dots,x_s)$, $A_3=(x_s,\dots,v)$, such that each arc of A_1 is an arc of C''', each arc of A_2 is an arc of E_1 and each arc of A_3 is an arc of P. This circle does not contain e_1 and a_1, a contradiction.

(2.2) We may assume that at least one of the vertices a_1, a_2 is a vertex of P. Let a_1 be a vertex of P. By condition (i) of Corollary 3 there is a circle C''' such that a_1 is not a vertex of C'''. Hence e_1 is an arc of C'''. Let v be a common vertex of C''' and P. Observe that C''' does not contain x_s and does not contain internal vertices of E_2. Hence g_1 is not an arc of C'''. Moreover, v is a vertex of path $(a_1,\dots,x_e)$ such that its arcs are arcs of P. Let path $F_1=(x_f,\dots,x_s)$ be the part of E_1 such that x_f is the only common vertex of C''' and E_1. Without loss of generality we may assume that for any circle C such that a_1 is not a vertex of C, common vertices of C and E_1 are not internal vertices of F_1. Let f_1 be an arc of F_1 incident with x_f. According to condition (ii) of Corollary 3 by b_1 we denote a vertex such that a digraph obtain from $\bar{D}$ by removing f_1 and b_1 does not contain any oriented circles. Arc f_1 is not an arc of C''' and then b_1 is a vertex of the part of P from a_1 to x_e. By condition (i) of Corollary 3 there is a circle C^{iv} such that b_1 is not a vertex of C^{iv}. Thus f_1 is an arc of C^{iv}. Vertex a_1 is a vertex of C^{iv} and b_1 is a vertex of C'''. It is clear that C^{iv} does not contain any internal vertices of E_2. Let $A_1=(x_f,\dots,b_1)$, $A_2=(a_1,\dots,x_f)$ be directed paths such that each arc of A_1 is an arc of C''' and each arc of A_2 is an arc of C^{iv}. If paths A_1 and A_2 have a common internal vertex then they create a circle C such that e_1 is not an arc of C and a_1 is not a vertex of C, a contradiction. Hence A_1 and A_2 do not have common internal vertices. Let us consider the following three circles: C'', a circle created by paths: A_1, $A_3=(b_1,\dots,x_f)$ such that each arc of A_3 is an arc of C', a circle created by paths: A_2, $A_4=(x_f,\dots,a_1)$ such that each arc of A_4 is an arc of C'. There is not a common vertex of these three circles. If there are any another arcs in $\bar{D}$ then using condition (ii) of Corollary 3 we obtain a contradiction. Otherwise C''' and C^{iv} have less vertices than the circle C' has, a contradiction.

Thus we obtain that $D \in LC_1$. $\square$

Unfortunately, we obtain much more complicated situation for $k \geqslant 2$.

We find a TT_n-maximal digraph of order m in two cases with very special assumptions.

Proposition 10. *Let D' be a digraph such that every circle of $\bar{D}'$ has at most two common vertices with the other circles of $\bar{D}'$. Let D be a TT_n-maximal digraph of order m such that D' can be obtained from D by a sequence of contractions of the second type. Under the assumption above, the minimum size of D is equal to $(2m-n)(n-2)$ and only digraphs on $m=n+k$ vertices of set LC_k are TT_n-maximal digraphs of order m and minimum size.*

Proof. Let D be a TT_n-maximal digraph of minimum size. By Proposition 7 we may assume that $D'=D$. We start with an observation.

Observation. Let C^1 and C^2 be different oriented circles of $\bar{D}$. Then C^1 and C^2 have at most one common vertex.

To obtain a contradiction, suppose that C^1 and C^2 have two common vertices v_1, v_2. Observe that every circle C such that C and C^i ($i=1$ or 2) have common vertex, contains at least one of vertices v_1, v_2. For an arc of circle C^1 by condition (ii) of Corollary 3 we find k vertices $w_1,\dots,w_k$. Observe that one of these vertices, w_1 for example, is a vertex of C^2. If $v_1 \in \{w_i,\ i=1,\dots,k\}$ or $v_2 \in \{w_i,\ i=1,\dots,k\}$ then we use condition (i) of Corollary 3 and obtain a contradiction. Let $v_j \notin \{w_i,\ i=1,\dots,k\}$, $j=1,2$. Let us consider a set $\{w_2,\dots,w_k,v_1\}$. We use condition (i) of Corollary 3 and obtain a contradiction.

By Propositions 6 and 4 we may assume that it is impossible to obtain a digraph from D by the contraction of the first type and that $\bar{D}$ does not contain isolated vertices. Observe that every circle of $\bar{D}$ has length 2. By Observation $\bar{D}$ consists of symmetric arcs. In this case we obtain digraph D of minimum size if D is a digraph obtained from K_n-maximal graph of the minimum size (Theorem 2) by replacing edges by symmetric arcs. Then $D \in LC_k$ because $\bar{D}$ does not contain isolated vertices. $\square$

Proposition 11. *Let D be a TT_n-maximal digraph of order $m=n+k$ and let there are $k+1$ independent vertices in D. Under the assumption above, the minimum size of D is equal to $(2m-n)(n-2)$ and only digraphs on $n+k$ vertices of set LCT_k are TT_n-maximal digraphs of order m and minimal size.*

Proof. Let D be a TT_n-maximal digraph of minimum size. By Propositions 6 and 7 we may assume that it is impossible to obtain a digraph from D by the contraction of the first or of the second type. In this way we simplify the describing of the structure of D. By Proposition 4 we can assume that $\bar{D}$ does not contain any isolated vertices.

In fact, we prove something more. Observe that it is sufficient to assume that there are $k+1$ vertices in $\bar{D}$ such that every two of them are joined by a symmetric arc in $\bar{D}$.

Let $v_1,\dots,v_{k+1}$ be independent vertices in D. By condition (i) of Corollary 3 there is a circle C^1 in $\bar{D}$ such that none of vertices v_i, $i=2,\dots,k+1$ is a vertex of C^1. Let us first suppose that v_1 is not a vertex of C^1. Observe that in this case C^1 is an isolated circle in $\bar{D}$. Digraph D is TT_n-maximal but does not have minimum size, a contradiction. We may assume that v_1 is a vertex of C^1. Similarly, for any $i=1,\dots,k+1$ there is a circle C^i such that v_i is a vertex of C^i and none of vertices v_j, $j\in\{1,\dots,k+1\}$, $j\neq i$ is a vertex of C^i. Observe that there are no more arcs in $\bar{D}$. Using condition (ii) of Corollary 3 we obtain that every two circles C^i, C^j, $i,j\in\{1,\dots,k+1\}$, $i\neq j$ have a common vertex. If two circles C^i, C^j, $i\neq j$ has at least two common disjoined paths then their arcs create another circles. We use condition (ii) of Corollary 3 and obtain a contradiction.

Hence every two different circles C^i $(i=1,\dots,k+1)$ have exactly one common path. Let us consider three different circles C^i, $i=1,\dots,k+1$. Without loss of generality we can take circles C^1, C^2 and C^3. Let P_{12} be a common path of C^1 and C^2 and let P_{23} be a common path of C^2 and C^3. Observe that P_{12} and P_{23} have a common vertex. Hence there is a vertex v such that it is a vertex of every circle C^i, $i=1,\dots,k+1$. Let P_{ij} be a common path of C^i and C^j, $i,j\in\{1,\dots,k+1\}$, $i\neq j$. The vertex v is a vertex of each P_{ij}. If all paths P_{ij} have length 0 then $D\in LC_k$. Otherwise we obtain $D\in LCT_k$. $\square$

Let us consider a digraph D obtained from K_n-maximal graph of the minimum size by replacing edges by symmetric arcs. Observe that a digraph whose complement is obtained from $\bar{D}$ by removing isolated vertices is a digraph of LC_k.

All the above-mentioned facts, propositions and theorems motivate us to state the following conjecture:

Conjecture 1. Let $m\geqslant n\geqslant 3$. The minimum size of a TT_n-maximal digraph of order m is equal to $(2m-n)(n-2)$. Only digraphs on $m=n+k$ vertices of LCT_k are TT_n-maximal digraphs of order m and minimum size.

It is surprised that if Conjecture 1 is true then the extremal digraph is not unique. It is different from analogous results for graphs and digraphs mentioned at the beginning. Moreover, there is a digraph D_1 of LC_k such that the addition of any arc to D_1 results in a digraph that does contain any tournament of order n. As an example we take D_1 with k independent vertices $v_1,\dots,v_k$. The remaining n vertices $u_1,\dots,u_n$ of D_1 are neighbours each other. Arcs $(u_i,u_{(i+1)\bmod n})$ $i=1,\dots,n$, create a directed circle, the remaining arcs joining vertices u_i $i=1,\dots,n$ are symmetric. Vertices v_j are not neighbours of vertices u_1, u_2, $j=1,\dots,k$. Vertices u_i, $i=3,\dots,n$

and v_j, $j=1,\dots,k$, are joined by a symmetric arcs. After the addition of any arc and the removing k proper vertices, antisymmetric arcs create a Hamiltonian path or a directed path with the last arc directed opposite. Such paths are contained by any tournament [7,1].

There is also a digraph $D_2 \in LC_k$ without the above-mentioned property. For example let D_2 be a digraph very similar to the above-described D_1. The only difference is that vertices v_j are not neighbours of vertices u_1, u_3. Vertices u_i, $i=2,4,\dots,n$ and v_j, $j=1,\dots,k$ are joined by symmetric arcs. We also assume that $n\geqslant 5$. After the addition of arc (u_1,v_1) we have to remove vertices v_j, $j=2,\dots,k$ and u_3 and obtain the digraph such that it does not contain the tournament with the following decreasing scores sequence of outdegrees of the vertices $(n-1,n-2,\dots,n-(n-3),1,1,1)$.

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